

GALERKIN METHOD FOR LINEAR INTEGRAL-ALGEBRAIC EQUATIONS OF INDEX 1

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ABSTRACT. In this paper, we study direct and indirect Galerkin method for solving linear Integral-Algebraic Equations of index 1. Convergence of indirect method is also analyzed.

1. INTRODUCTION

Recently numerical solution for Integral-Algebraic Equation (IAE) has been studied by many researchers [1]. In physical models with constrained time or space variables, Algebraic Equations including Integral or Differential Equations arise. Therefore, the investigation of mixed types of equations is important. This can be seen by observing the number of papers that are about Differential-Algebraic Equations, Integro Differential Equations, and so on. One of these systems which mixes Volterra Integral Equations of first and second type is the system of Integral-Algebraic Equations. However, we don't find any application for this systems except [6, 7, 8, 9], but it seems that they be very important in the future. In this paper we use the standard Galerkin method, for the system of Integral Algebraic Equations of index 1. We implement the Galerkin method by considering the structure of IAE of index 1 in two ways, direct and indirect and we analyze indirect method here. We Consider the system

$$(1.1) \quad A(t)y(t) + \int_0^t k(t, s)y(s)ds = f(t), \quad t \in I := [0, T],$$

where $A \in \mathbf{C}(I, \mathbb{R}^{r \times r})$, $f \in \mathbf{C}(I, \mathbb{R}^r)$ and $k \in \mathbf{C}(\mathbb{D}, \mathbb{R}^{r \times r})$ with $\mathbb{D} := \{(t, s) : 0 \leq s \leq t \leq T\}$. If $A(t)$ is nonsingular for all $t \in I$ then multiplying (1.1) by A^{-1} changes it to a system of the second kind Volterra Integral Equations that it's theoretical and numerical analysis have been almost completely investigated [3, 4]. Otherwise, if $A(t)$ is a singular matrix with constant rank for all $t \in I$, then, the system (1.1) is an IAE. The classification of this system done by introducing index notation. The differential index of differential-algebraic equation can be extended to this system. The following extension is done by Gear in [5].

Definition 1.1. We say the system (1.1) has index m if, m is the minimum possible number of differentiating (1.1) to obtain a system of second kind Volterra Integral equation.

For example the following system

$$(1.2) \quad x(t) = f_1(t) + \int_0^t k^{11}(t, s)x(s) + k^{12}(t, s)y(s)ds,$$

Key words and phrases. Galerkin method, Differential Index, Integral-Algebraic Equations.

$$(1.3) \quad 0 = f_2(t) + \int_0^t k^{21}(t, s)x(s) + k^{22}(t, s)y(s)ds.$$

with the functions $f_i, k^{ij}, i, j = 1, 2$ are sufficiently smooth and $f_2(0) = 0, |k^{22}(t, t)| \geq k_0 > 0$ for all $t \in [0, T]$; (1.2) and (1.3) is the system of Integral-Algebraic Equations of index 1. For this system the Piecewise Polynomial Collocation method is investigated by Kauthen in [2]. In this paper we investigate the Galerkin method for this system.

2. PRELIMINARY

The Galerkin method is a projection type method [3, 4] that we apply to IAE of index 1. Let $\mathcal{K} : \mathbf{X} \rightarrow \mathbf{X}$ be an operator, where \mathbf{X} is a Hilbert space with orthonormal base $\{\phi_i\}_{i=1}^\infty$ and \mathbf{X}_n be an n -dimensional subspace of \mathbf{X} generated by $\{\phi_i\}_{i=1}^n$. Then the system (1.2) and (1.3) changes to

$$(2.1) \quad \mathcal{K}x = f.$$

For using the Galerkin method's we define the projection $P_n : \mathbf{X} \rightarrow \mathbf{X}_n$ by $P_n x = \sum_{i=1}^n \langle x, \phi_i \rangle \phi_i$. Then it is clear that $\|P_n x - x\| \rightarrow 0$ as $n \rightarrow \infty$. We search $\mathbf{x}_n \in \mathbf{X}_n, \mathbf{x}_n = \bar{x}\bar{\phi}$ with $\bar{x} = (x_1, \dots, x_n)$ and $\bar{\phi} = (\phi_1, \dots, \phi_n)^T$, such that

$$(2.2) \quad P_n \mathcal{K} \mathbf{x}_n = P_n f.$$

The corresponding operator of the system (1.1) is defined by

$$(2.3) \quad \mathcal{K}([x, y]) = \left(x - \int_0^t k^{11}(t, s)x(s) + k^{12}(t, s)y(s)ds, - \int_0^t k^{21}(t, s)x(s) + k^{22}(t, s)y(s)ds \right)^T,$$

and the assumption that $f = (f_1, f_2)^T, \mathbf{X} \subseteq (\mathcal{L}^2[0, T])$ and the functions $f_i, k^{ij}, i, j = 1, 2$ are sufficiently smooth and $f_2(0) = 0, |k^{22}(t, t)| \geq k_0 > 0$ for all $t \in [0, T]$, guarantee us that $\mathcal{K}x = f$ has a continuous solution as it follows. We first differentiate equation (1.3) w. r. to t and we have

$$0 = f_2'(t) + k^{21}(t, t)x(t) + k^{22}(t, t)y(t) + \int_0^t \frac{\partial k^{21}(t, s)}{\partial t} x(s) + \frac{\partial k^{22}(t, s)}{\partial t} y(s)ds.$$

Because $|k^{22}(t, t)| \geq k_0 > 0$ we obtain

$$(2.4) \quad y(t) = -\frac{f_2'(t)}{k^{22}(t, t)} - \frac{k^{21}(t, t)}{k^{22}(t, t)}x(t) - \int_0^t \frac{1}{k^{22}(t, t)} \frac{\partial k^{21}(t, s)}{\partial t} x(s) - \frac{1}{k^{22}(t, t)} \frac{\partial k^{22}(t, s)}{\partial t} y(s)ds,$$

as a Volterra integral equations of the second kind for $y(t)$. Now it is easy to show that this system has a unique solution (see [4]).

3. IMPLEMENTATION AND CONVERGENCE

We can implement the Galerkin method for two systems (1.2), (1.3) and (1.2), (2.4). For the former the Galerkin method is called direct and for the further, is called indirect method. For the second system, the convergence and order of convergence can be obtained by similar arguments as discussed [6] and [3]. In [3] the order of convergence has been explained for Fredholm integral equation. Since, by using

suitable kernel, each Volterra integral equation can be changed to a Fredholm integral equation, Galerkin method for both are the same. Hence the convergence order of the approximate solution is equal to the convergence order of the best approximation in space X_n . This is confirmed by numerical examples. Let $\{V_i\}_{i=1}^\infty$ be Legendre polynomials as orthonormal base for $\mathcal{L}^2[-1, 1]$. For $\mathcal{L}^2[0, T]$ we use shifted Legendre polynomials $\{V_{T_i}\}_{i=1}^\infty$, $V_{T_i}(s) = \sqrt{\frac{2}{T}} V_i(\frac{2*s-T}{T})$, $s \in [0, T]$ and for $(\mathcal{L}^2[0, T])^2$, we use $\{(V_{T_i}, 0), (0, V_{T_i})\}_{i=1}^\infty$ with $\langle (x, y), (z, t) \rangle = \langle x, z \rangle_2 + \langle y, t \rangle_2$ and $\langle x, y \rangle_2 = \int_0^T x \bar{y} dt$. Also we suppose $P_n([x, y]) = (\sum_{i=1}^n x_i V_{T_i}, \sum_{i=1}^n y_i V_{T_i})$. For equations (1.2), (1.3) and (2.4) we have.

$$(3.1) \quad \begin{aligned} \sum_{i=1}^n x_i V_{T_i}(t) = & f_1(t) + \sum_{i=1}^n x_i \int_0^t k^{11}(t, s) V_{T_i}(s) ds \\ & + \sum_{i=1}^n y_i \int_0^t k^{12}(t, s) V_{T_i}(s) ds, \end{aligned}$$

$$(3.2) \quad 0 = f_2(t) + \sum_{i=1}^n x_i \int_0^t k^{21}(t, s) V_{T_i}(s) ds + \sum_{i=1}^n y_i \int_0^t k^{22}(t, s) V_{T_i}(s) ds,$$

$$(3.3) \quad \begin{aligned} \sum_{i=1}^n y_i V_{T_i}(t) = & -\frac{f_2'(t)}{k^{22}(t, t)} - \frac{k^{21}(t, t)}{k^{22}(t, t)} \sum_{i=1}^n x_i V_{T_i}(t) \\ & - \sum_{i=1}^n x_i \int_0^t \frac{1}{k^{22}(t, t)} \frac{\partial k^{21}(t, s)}{\partial t} V_{T_i}(s) ds \\ & - \sum_{i=1}^n y_i \int_0^t \frac{1}{k^{22}(t, t)} \frac{\partial k^{22}(t, s)}{\partial t} V_{T_i}(s) ds. \end{aligned}$$

Now we can use (3.1), (3.2) for the first system and (3.1), (3.3) for the second system introduced above. It is important for analysis to note that formula (3.3) is derivative of (3.2). Now if we multiply equations (3.1)-(3.3) to $V_{T_j}(t)$ and integrate on the $[0, T]$, we have obtained

$$(3.4) \quad \begin{aligned} x_j = & \int_0^T f_1(t) V_{T_j}(t) dt + \sum_{i=1}^n x_i \int_0^T \int_0^t k^{11}(t, s) V_{T_i}(s) V_{T_j}(t) ds dt \\ & + \sum_{i=1}^n y_i \int_0^T \int_0^t k^{12}(t, s) V_{T_i}(s) V_{T_j}(t) ds dt, \end{aligned}$$

$$(3.5) \quad \begin{aligned} 0 = & \int_0^T f_2(t) V_{T_j}(t) dt + \sum_{i=1}^n x_i \int_0^T \int_0^t k^{21}(t, s) V_{T_i}(s) V_{T_j}(t) ds dt \\ & + \sum_{i=1}^n y_i \int_0^T \int_0^t k^{22}(t, s) V_{T_i}(s) V_{T_j}(t) ds dt, \end{aligned}$$

$$\begin{aligned}
(3.6) \quad y_j = & - \int_0^T \frac{f_2'(t)}{k^{22}(t,t)} dt - \sum_{i=1}^n x_i \int_0^T \int_0^t \frac{1}{k^{22}(t,t)} \frac{\partial k^{21}(t,s)}{\partial t} V_{T_i}(s) V_{T_j}(t) ds dt \\
& - \sum_{i=1}^n x_i \int_0^T \frac{k^{21}(t,t)}{k^{22}(t,t)} V_{T_i}(t) V_{T_j}(t) dt \\
& - \sum_{i=1}^n y_i \int_0^T \int_0^t \frac{1}{k^{22}(t,t)} \frac{\partial k^{22}(t,s)}{\partial t} V_{T_i}(s) V_{T_j}(t) ds dt.
\end{aligned}$$

The direct and indirect Galerkin method for system (1.2), (1.3) are solving the linear systems (3.4), (3.5) and (3.4), (3.6) respectively.

Lemma 3.1. [4] *Let V and W be normed spaces, with W complete. Let $K \in \mathcal{L}(V, W)$, let $\{K_n\}$ be a sequence of compact operators in $\mathcal{L}(V, W)$, and assume $K_n \rightarrow K$ in $\mathcal{L}(V, W)$. Then K is compact.*

Lemma 3.2. [4] *Let V be a Banach space, and let $\{P_n\}$ be a family of bounded projections on V with*

$$P_n u \rightarrow u \quad \text{as } n \rightarrow \infty, u \in V$$

If $K : V \rightarrow V$ is compact, then

$$\|K - P_n K\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Theorem 3.3. [4] *Assume $K : V \rightarrow V$ is bounded, with V a Banach space; and assume $\lambda - K : V \rightarrow V$ is one to one and surjective. Further assume*

$$\|K - P_n K\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then for all sufficiently large n , say, $n \geq N$, the operator $(\lambda - K)^{-1}$ exists as a bounded operator from V to V . Moreover, it is uniformly bounded:

$$\sup_{n \geq N} \|(\lambda - P_n K)^{-1}\| \leq \infty.$$

For the solutions u_n (n sufficiently large) of $P_n(\lambda - K)u_n = P_n F$ and the solutions u of $(\lambda - K)u = F$, we have

$$u - u_n = \lambda(\lambda - P_n K)^{-1}(u - P_n u)$$

and the two-sided error estimate

$$\frac{|\lambda|}{\|\lambda - P_n K\|} \|u - P_n u\| \leq \|u - u_n\| \leq |\lambda| \|(\lambda - P_n K)^{-1}\| \|u - P_n u\|.$$

We have following Theorem for indirect method.

Theorem 3.4. *Let for the system (1.2) and (1.3) the following conditions are satisfied:*

- (1) $f_i \in C^1(0, T)$, $K^{i,j}[t, s] \in C^1(\mathbb{D})$, for all $i, j = \{1, 2\}$,
- (2) $k^{22}(t, t) \geq k_0 > 0$,

where $\mathbb{D} := \{(t, s) : 0 \leq s \leq t \leq T\}$. Then the convergence order of the approximate solution is equal to the convergence order of the best approximation in the space $X_n = \{V_{T_i}\}_{i=1}^\infty$.

Proof. The space $\mathcal{L}^2[0, T]$ with \mathcal{L}^2 Norm is Banach space, hence the Cartesian product on this space, i.e., $V = (\mathcal{L}^2[0, T])^2$, with introduced norm generate also Banach space. Now we change the system (1.2) and (2.4) to the following system

$$(3.7) \quad (I - K)x = F, \quad F = [F_1, F_2]^T, \quad x = [x_1, x_2]^T,$$

with

$$\begin{aligned} F_1(t) &= f_1(t), & F_2(t) &= -\frac{f_2'(t)}{k^{22}(t, t)} - \frac{k^{21}(t, t)}{k^{22}(t, t)} f_1(t), \\ K^{11}(t, s) &= \begin{cases} k^{11}(t, s), & s \leq t, \\ 0, & s > t, \end{cases} & K^{12}(t, s) &= \begin{cases} k^{12}(t, s), & s \leq t, \\ 0, & s > t, \end{cases} \\ K^{21}(t, s) &= \begin{cases} -\frac{k^{21}(t, t)}{k^{22}(t, t)} k^{11}(t, s) - \frac{1}{k^{22}(t, t)} \frac{\partial k^{21}(t, s)}{\partial t}, & s \leq t, \\ 0, & s > t, \end{cases} \\ K^{22}(t, s) &= \begin{cases} -\frac{k^{21}(t, t)}{k^{22}(t, t)} k^{12}(t, s) - \frac{1}{k^{22}(t, t)} \frac{\partial k^{22}(t, s)}{\partial t}, & s \leq t, \\ 0, & s > t, \end{cases} \end{aligned}$$

and with the integral operator

$$Kx = \int_0^T \begin{bmatrix} K^{11}(t, s) & K^{12}(t, s) \\ K^{21}(t, s) & K^{22}(t, s) \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} ds.$$

By introducing

$$K_n x := \int_0^T \begin{bmatrix} K_n^{11}(t, s) & K_n^{12}(t, s) \\ K_n^{21}(t, s) & K_n^{22}(t, s) \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} ds,$$

with continuous kernels

$$\begin{aligned} K_n^{11}(t, s) &= \begin{cases} k^{11}(t, s), & s \leq t, \\ -n(k^{11}(t, t))((t + \frac{1}{n}) - s), & s \in [t, t + \frac{1}{n}], \\ 0, & s > t + \frac{1}{n}, \end{cases} \\ K_n^{12}(t, s) &= \begin{cases} k_n^{12}(t, s), & s \leq t, \\ -n(k^{12}(t, t))((t + \frac{1}{n}) - s), & s \in [t, t + \frac{1}{n}], \\ 0, & s > t + \frac{1}{n}, \end{cases} \\ K_n^{21}(t, s) &= \begin{cases} -\frac{k^{21}(t, t)}{k^{22}(t, t)} k^{11}(t, s) - \frac{1}{k^{22}(t, t)} \frac{\partial k^{21}(t, s)}{\partial t}, & s \leq t, \\ -n(\frac{k^{21}(t, t)}{k^{22}(t, t)} k^{11}(t, t) + \frac{1}{k^{22}(t, t)} \frac{\partial k^{21}(t, t)}{\partial t})((t + \frac{1}{n}) - s), & s \in [t, t + \frac{1}{n}], \\ 0, & s > t + \frac{1}{n}, \end{cases} \\ K_n^{22}(t, s) &= \begin{cases} -\frac{k^{21}(t, t)}{k^{22}(t, t)} k^{12}(t, s) - \frac{1}{k^{22}(t, t)} \frac{\partial k^{22}(t, s)}{\partial t}, & s \leq t, \\ -n(\frac{k^{21}(t, t)}{k^{22}(t, t)} k^{12}(t, t) + \frac{1}{k^{22}(t, t)} \frac{\partial k^{22}(t, t)}{\partial t})((t + \frac{1}{n}) - s), & s \in [t, t + \frac{1}{n}], \\ 0, & s > t + \frac{1}{n}. \end{cases} \end{aligned}$$

which is compact and using lemma 3.1, imply K is compact.

Applying the Galerkin method to the Fredholm system (3.7) leads to the linear system equivalent with the system (3.4), (3.6). So it is enough to show that for this system the convergence order of the approximate solution by the Galerkin method is equal to the convergence order of the best approximation. Since, the system (3.7) also is a volterra system, using the same line of [4], $(I - K)$ is invertible so is surjective. The Lemma 3.2 provide the remainder condition. Finally all conditions of the theorem 3.3 are satisfied and this completes the proof. \square

TABLE 1. Numerical result obtained from (3.4),(3.5)

n	2	4	6	8	10
$\ x_n - x\ $	$4.0e-002$	$3.0e-004$	$7.4e-007$	$9.4e-010$	$7.6e-013$
$\ y_n - y\ $	$1.6e-001$	$2.2e-003$	$8.1e-006$	$1.4e-008$	$1.4e-011$

TABLE 2. Numerical result obtained from (3.4),(3.6)

n	2	4	6	8	10
$\ x_n - x\ $	$3.8e-002$	$2.6e-004$	$6.7e-007$	$8.8e-010$	$6.8e-013$
$\ y_n - y\ $	$7.3e-002$	$5.1e-004$	$1.3e-006$	$1.7e-009$	$1.3e-012$

TABLE 3. Logarithm of the error by best approximation using Legendre polynomial)

n	2	4	6	8	10
$\ P_n \sin(t) - \sin(t)\ $	$5.1e-002$	$3.3e-004$	$8.1e-007$	$1.0e-009$	$8.0e-013$
$\ P_n \cos(t) - \cos(t)\ $	$7.8e-002$	$5.4e-004$	$1.4e-006$	$1.7e-009$	$1.3e-012$

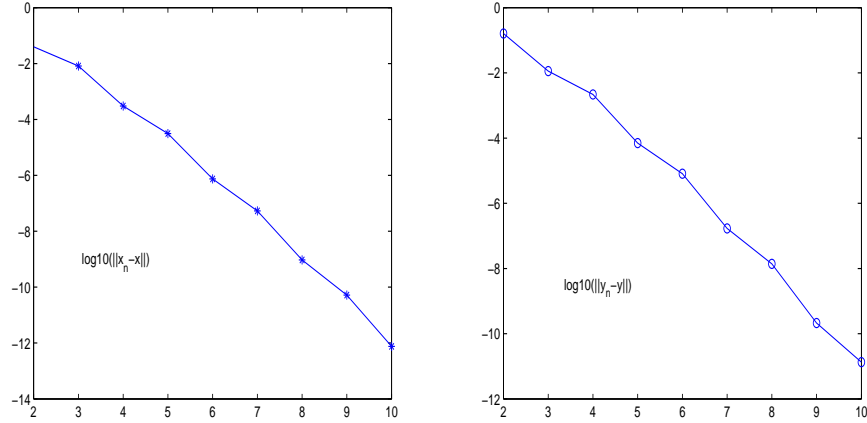


FIGURE 1. plots of convergence order by direct Galerkin method

4. NUMERICAL RESULTS

In this section we illustrate the methods by numerical examples.

Example 1.

We consider (2.3) with $k^{11} = s + t$, $k^{12} = s^2 + t^2$, $k^{21} = s - t^2$, $k^{31} = s + t + 1$, $f_1 = -t - 2\sin(t)t^2 + 2\sin(t)$ and $f_2 = t^2 - 2\sin(t) + \cos(t)t - \cos(t)t^2 + 1 - \cos(t) - 2\sin(t)t$

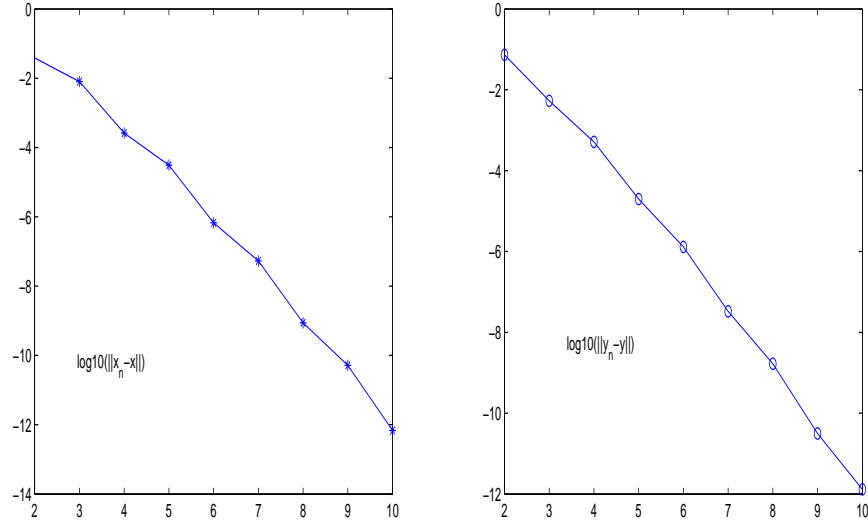


FIGURE 2. plots of convergence order by indirect Galerkin method. We note that the slope of this figures is same as the slope of figure 3 as claimed

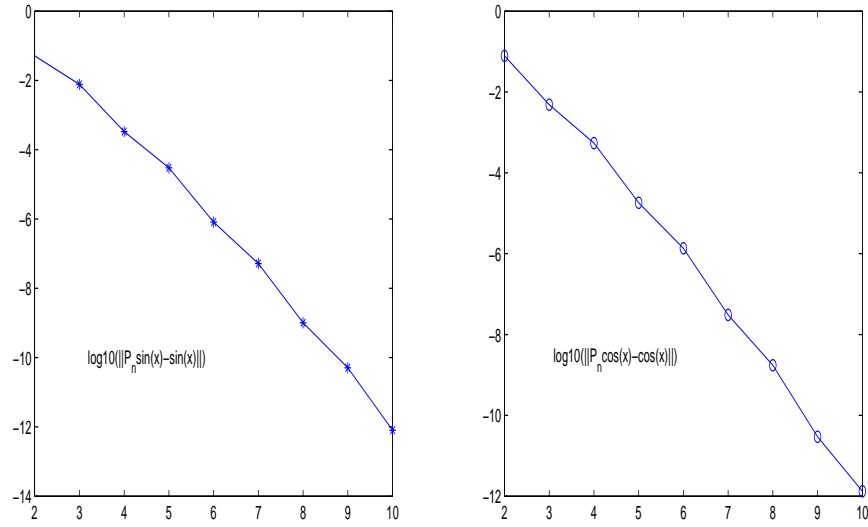


FIGURE 3. plots of convergence order of best the approximation

where the exact solutions are $x = \sin(t)$ and $y = \cos(t)$. Numerical results obtained by two methods are in table 1 and 2 which show rapid convergence. In these tables $\|f\| = \max_{T \in [0, T]} |f(t)|$. In the following figures, we also illustrate $\log_{10}(\|x_n - x\|)$

and $\log_{10}(\|y_n - y\|)$. For the second system, we expect that this figure to be similar to the figure of $\log_{10}(\|P_n x - x\|)$ and $\log_{10}(\|P_n y - y\|)$. This numerical experiment shows that the second system is more accurate than the first. But we need more computations in comparison with the first.

Remark 4.1. In equations (3.4), (3.5) and (3.6), we find two type of integrals that should be computed, single and double integrals. The integral of type $\int_0^T f(t)dt$ can be computed by Gaussian Integration Methods (see [11]) of the form

$$\int_{-1}^1 f(t)dt \simeq \sum_{i=1}^n w_i f(x_i)$$

and by changing of variable we have

$$(4.1) \quad \int_0^T f(t)dt \simeq \sum_{i=1}^n w_i f(Tx_i/2 + T/2)$$

where x_1, \dots, x_n are the roots of $V_{T(n+1)}$, and w_1, \dots, w_n are computed by the system expressed in (3.6.13) of [11]. An approximate values of are computed as follows $\int_0^T \int_0^t f(t, s)dsdt$ in this calculation is as it follows

$$(4.2) \quad \int_0^T \int_0^t f(t, s)dsdt = \sum_{i=1}^n \sum_{j=1}^n w_i w_j f\left(\frac{Tx_i}{2} + \frac{T}{2}, \frac{(\frac{Tx_i}{2} + \frac{T}{2})x_i}{2} + \frac{\frac{Tx_i}{2} + \frac{T}{2}}{2}\right)$$

Also we notice that the term n is not fixed for different P_m , indeed in our calculation $n = m$ that means for improving accuracy of the solution we improve the accurate approximate solutions of integrals.

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